CHERENKOV RADIATION IN A SEMI-INFINITE DIELECTRIC MEDIUM WITH A CONDUCTING BOUNDARY

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ABSTRACT. Cherenkov radiation due to the passage of a point charge moving parallel to the conducting boundary inside a semi-infinite dielectric medium has been studied. The formulae obtained are in agreement with the formulae for Cherenkov radiation of electron in an infinite homogeneous medium in the limiting case.

INTRODUCTION

Linhart (1955) and Danos (1955) worked out the problem of Cherenkov radiation emitted by an electron moving in vacuum with uniform motion on a straight line parallel to the plane face of semi-infinite dielectric medium. Here we propose to calculate electro-magnetic field and Cherenkov radiation due to a charge particle moving inside a semi-infinite dielectric medium with a constant velocity on a straight line parallel to the plane boundary. The other side is a semi-infinite conducting medium. The conductivity is assumed to be high and in fact taken to be infinite for simplicity of calculation.

STATEMENT OF PROBLEM

Let $x = 0$ be the equation of the boundary surface; $x > 0$, the dielectric medium and $x < 0$, the conducting medium. A particle of charge e is moving with a constant velocity v in a straight line parallel to z-axis at a distance a from the boundary surface. We are interested in the case when v is greater than the phase velocity of light in the dielectric medium.

Maxwell's equations for field variables E and H are

$$\left.\begin{aligned}
\text{rot } H_\omega &= \frac{iw}{c} E_\omega + \frac{4\pi}{c} j_\omega \\
\text{rot } E_\omega &= -\frac{iw}{c} H_\omega \\
\text{div } E_\omega &= \frac{4\pi}{\epsilon} \rho_\omega \\
\text{div } H_\omega &= 0,
\end{aligned}\right\} \quad \ldots (1)$$

where j is the current density, ρ is the density of free charges and ϵ is the dielectric constant. All quantities are used as Fourier Transform. Introducing vector and scalar potential A and ϕ we arrive at the following set :

$$\left.\begin{aligned} &H_\omega = \text{rot } A_\omega \\ &E_\omega = -\frac{i\omega}{c} A_\omega - \text{grad } \phi_\omega \\ &\nabla^2 A_\omega + \frac{\epsilon\omega^2}{c^2} A_\omega = -\frac{4\pi}{c} j_\omega \\ &\text{with the condition div } A_\omega + \frac{i\epsilon\omega}{c} \phi_\omega = 0 \end{aligned}\right\} \quad \ldots (2)$$

Here $j_x = 0 = j_y$ and $j_z = ev\delta(x-a)\,\delta(y)\,\delta(z-vt)$ ... (3)

SOLUTION

Taking $A_x = 0 = A_y$,

$$\nabla^2 A_z + \frac{\epsilon\omega^2}{c^2} A_z = -\frac{2e}{c}\delta(x-a)\delta(y)e^{-i\omega\frac{z}{v}} \quad \ldots (4)$$

To solve the equation (4) we assume that

$$A_z = u(x, y)e^{-\frac{i\omega z}{v}}.$$

From (4) we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + s^2 u = -\frac{2e}{c}\delta(x-a)\delta(y), \quad \ldots (5)$$

where
$$s^2 = \frac{w^2}{v^2}(\epsilon\beta^2 - 1), \; \beta = \frac{v}{c}.$$

There is no electromagnetic field inside the conducting medium. On the boundary surface tangential components of E and normal components of H should be zero. To satisfy these conditions u and $\partial u/\partial y$ should be zero at $x = 0$. Besides there is a singularity at $x = a$, $y = 0$. Thus the solution of (5) is

$$u = -\frac{ie}{2c}[H_0^{(2)}(sq_1) - H_0^{(2)}(sq_2)], \quad \ldots (6)$$

where
$$q_1^2 = (x-a)^2 + y^2, \quad q_2^2 = (x+a)^2 + y^2.$$

$$\left.\begin{aligned} \therefore\; A_z(\omega) &= -\frac{ie}{2c}[H_0^{(2)}(sq_1) - H_0^{(2)}(sq_2)]e^{-i\omega\left(t-\frac{z}{v}\right)} \\ \text{and} \quad \phi(\omega) &= -\frac{ie}{2v\epsilon}[H_0^{(2)}(sq_1) - H_0^{(2)}(sq_2)]e^{-i\omega\left(t-\frac{z}{v}\right)} \end{aligned}\right\} \quad \ldots (7)$$

The field components are

$$
\left.
\begin{aligned}
H_x(w) &= \frac{ies}{2c}\left[\frac{y}{q_1}H_1^{(2)}(sq_1)-\frac{y}{q_2}H_1^{(2)}(sq_2)\right]e^{i\omega\left(t-\frac{z}{v}\right)}\\
H_y(\omega) &= -\frac{-ies}{2c}\left[\frac{x-a}{q_1}H_1^{(2)}(sq_1)-\frac{x+a}{q_2}H_1^{(2)}(sq_2)\right]e^{i\omega\left(t-\frac{z}{v}\right)}\\
E_x(\omega) &= -\frac{-ies}{2v\epsilon}\left[\frac{x-a}{q_1}H_1^{(2)}(sq_1)-\frac{x+a}{q_2}H_1^{(2)}(sq_2)\right]e^{i\omega\left(t-\frac{z}{v}\right)}\\
E_y(\omega) &= \frac{-ies}{2v\epsilon}\left[\frac{y}{q_1}H_1^{(2)}(sq_1)-\frac{y}{q_2}H_1^{(2)}(sq_2)\right]e^{i\omega\left(t-\frac{z}{v}\right)}\\
E_z(\omega) &= -\left[\frac{es^2}{2\omega\epsilon}[H_0^{(2)}(sq_1)-H_0^{(2)}(sq_2)]\right]e^{i\omega\left(t-\frac{z}{v}\right)}
\end{aligned}
\right\} \quad \ldots \quad (8)
$$

One can easily verify that tangential components E_y and E_z and normal component H_x are zero at $x = 0$.

Surface charge and surface current : One notes that the normal component E_x and tangential component H produce surface charge and surface current on the boundary surface. If Σ and K are the surface charge density and surface current density then

$$
4\pi\Sigma = Re(\epsilon E_x)_{x=0} = -\frac{ea}{v}\int\frac{s}{q}\left[J_1(sq)\sin\omega\left(t-\frac{z}{v}\right)+N_1(sq)\cos\omega\left(t-\frac{z}{v}\right)\right]dw \quad \ldots \quad (9)
$$

$$
4\pi K = Re(H_y)_{x=0} = -ea\int\frac{s}{q}\left[J_1(sq)\sin\omega\left(t-\frac{z}{v}\right)+N_1(sq)\cos\omega\left(t-\frac{z}{v}\right)\right]dw
$$

where $q^2 = a^2+y^2$. ... (10)

Equation (9) shows that $\Sigma\to 0$ and $a\to 0$, i.e. there will be no surface charge when the point charge moves very closely to the boundary surface. This may be understood if we imagine a particle of opposite charge at the image point of the real particle similar to the static case. If the distance between these two oppositely charged particles is very small, the medium is not effectively polarized due to equal amount of polarization by the oppositely charged particles.

Calculation of Radiation : We now calculate the total energy radiated by the point charge through the surface of a half cylinder of large radius with the axis as z-axis. In cylindrical co-ordinates (r, θ, z)

$$
\begin{aligned}
H_\theta &= -H_x\sin\theta+H_y\cos\theta\\
&= \frac{-ies}{2c}\left[H_1^{(2)}(sq_1)\frac{r-a\cos\theta}{q_1}-H_1^{(2)}(sq_2)\frac{r+a\cos\theta}{q_2}\right]e^{i\omega\left(t-\frac{z}{v}\right)}
\end{aligned}
\quad \ldots \quad (11)
$$

where $q_1^2 = r^2-2ar\cos\theta+a^2$, $q_2^2 = r^2+2ar\cos\theta+a^2$.

For large values of r,

$$\left.\begin{aligned} H_\theta(\omega) &= \frac{es}{2c}\sqrt{\frac{2}{\pi sr}}\left[e^{-is(r-a\cos\theta)+i\frac{\pi}{4}} - e^{-is(r+a\cos\theta)+i\frac{\pi}{4}}\right]e^{i\omega\left(t-\frac{z}{v}\right)} \\ E_z(\omega) &= -\frac{es^2}{2\omega\epsilon}\sqrt{\frac{2}{\pi sr}}\left[e^{-is(r-a\cos\theta)+i\frac{\pi}{4}} - e^{-is(r+a\cos\theta)+i\frac{\pi}{4}}\right]e^{i\omega\left(t-\frac{z}{v}\right)} \end{aligned}\right\} \quad (12)$$

Since $s > 0$, waves (partly reflected at the boundary) are propagated at a large distance from the boundary surface.

$$\left.\begin{aligned} ReH_\theta &= \int_0^\infty \frac{2es}{c}\sqrt{\frac{2}{\pi sr}} \sin\omega\left(t-\frac{z}{v}-\frac{sr}{\omega}+\frac{\pi}{4w}\right)\sin(sa\cos\theta)d\omega \\ ReE_z &= -\int_0^\infty \frac{2es}{\epsilon\omega}\sqrt{\frac{2}{\pi sr}} \sin\omega\left(t-\frac{z}{v}-\frac{sr}{\omega}+\frac{\pi}{4w}\right)\sin(sa\cos\theta)dw \end{aligned}\right\} \quad \ldots \quad (13)$$

Radiation through the surface per unit length is

$$\frac{dW}{dl} = \frac{c}{4\pi}\int_{t=-\infty}^{\infty}\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}(-R_eH_\theta.R_0E_z)r\,d\theta\,dt$$

$$= \frac{e^2}{c^2}\int_0^\infty\left(1-\frac{1}{\epsilon\beta^2}\right)\omega[1-J_0(2sa)]d\omega \quad \ldots \quad (14)$$

In practice $$\frac{dW}{dl} = \frac{e^2}{c^2}\int_0^{\omega_{max}}\left(1-\frac{1}{\epsilon\beta^2}\right)\omega[1-J_0(2sa)]dw \quad \ldots \quad (15)$$

DISCUSSION

Energy loss per unit frequency interval per unit length is given by

$$S = \frac{e^2}{c^2}\left(1-\frac{1}{\epsilon\beta^2}\right)\omega[1-J_0(2sa)] \quad \ldots \quad (16)$$

In the limit when $a \to \infty$ the equation (16) should go to the expression of the homogeneous medium.

In infinite homogeneous dielectric medium Cherenkov radiation per unit frequency interval per unit path length is

$$S' = \frac{e^2}{c^2}\left(1-\frac{1}{\epsilon\beta^2}\right)\omega \quad \ldots \quad (17)$$

which obviously is the limit of S when $a \to \infty$.

When $a \to 0$, $S \to 0$, i.e. Cherenkov radiation does not take place, if the point charge moves closely to the boundary surface. It is due to the effect of a particle of opposite charge at the image point of the point charge.

If the distance of the particle from the boundary surface is increased, $J_0(2sa) < 1$ until the first zero of J_0 is obtained. In this case $S < S'$. Though the boun-fary surface reflects the waves, the point charge at the image point which is effectively the surface charge polarizes the medium in opposite direction and ultimately the total energy loss becomes less than the ordinary Cherenkov radiation. Thus in this situation the conducting barrier reduces the radiation. After the first zero of J_0, $J_0 < 0$ until the second zero is obtained. In this case $S > S'$ i.e. by suitably adjusting the distance of the particle from the boundeary surface the energy loss can be increased and it may be greater than ordinary Cherenkov radiation. Considering all these results we conclude that S oscillates with a and ultimately it converges to S'. A similar phenomenon through not exactly identical with the previous case occurs when variable of radiation intensty are considered as a function of w. These variations of intensity with a are to be traced to the phenomenon of interference.

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